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On the Solutions of the Lucas Sequence Equation $\pm\frac{1}{V_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k}$

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Abstract

Suppose that $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ are respectively the Lucas sequences of the first and second kinds with $P \neq 0$, $Q \neq 0$ and $gcd(P,Q) = 1$. In this paper, we introduce an approach for studying the solutions (x,n) of the diophantine equation

$$\pm\frac{1}{V_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k},$$

in the cases of $(P_1,Q_1) \neq (P_2,Q_2)$ and $(P_1,Q_1) = (P_2,Q_2)$. Moreover, we apply the procedure of this approach with which $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$ and $-1 \leq Q_2 \leq 1$. Our approach is mainly based on transferring this equation into either an elliptic curve equation that can be solved easily using the Magma software, or a quadratic equation that can be solved using the quadratic formula.

Keywords: Lucas sequences; diophantine equations; elliptic curves; quadratic equation.

1 Introduction

A diophantine equation is defined as n-variable function of the form $g(t_1, t_2, ..., t_n) = 0$ with $n \geq 2$. The variables $t_1, t_2, ..., t_n$ are generally required to be integers. The name is in honor of the Greek mathematician Diophantus, who wrote a treatise on the subject entitled Arithmetical. He lived in Egypt around 300 AD, died around the age of 84, and wrote 13 books of Arithmetica on the subject of which only 6 remain today. In his works, he stated the concept of diophantine equations and one of the oldest diophantine equations is Fermat's equation, which is defined by

$$x^n + y^n = z^n,$$

where x, y, z and n are positive integers. For more details about diophantine equations, one can see e.g.[4] and [2].

On the other hand, the sequence $\{G_n\}$ is called a linear recurrence sequence relation of order r if the recurrence

$$G_{n+r} = a_1 G_{n+r-1} + a_2 G_{n+r-2} + ... + a_r G_r,$$

is satisfied $\forall n \geq 0$ with the coefficients $(a_r \neq 0) \in \mathbb{C}$. As an example of the linear recurrence sequence is the binary recurrence sequence is represented as a sequence with the order $r = 2$. Also, there are many types of binary recurrence sequences such as the Lucas sequences of the first kind denoted by $\{U_n(P, Q)\}$ and the second kind denoted by $\{V_n(P, Q)\}$, which are respectively defined by

$$U_0 = 0, U_1 = 1, \quad U_{n+2} = PU_{n+1} - QU_n \quad \text{for} \quad n \geq 0, \tag{1}$$

$$V_0 = 2, V_1 = P, \quad V_{n+2} = PV_{n+1} - QV_n \quad \text{for} \quad n \geq 0, \tag{2}$$

where $P \neq 0$ and $Q \neq 0$ with $gcd(P, Q) = 1$. The Lucas sequences for certain P and Q give some well known sequences such as Fibonacci sequence, Lucas sequence, Pell sequence, and Pell-Lucas sequence, that are denoted as follows:

$$F_n = U_n(1, -1), L_n = V_n(1, -1), P_n = U_n(2, -1), \text{ and } Q_n = V_n(2, -1),$$

respectively. It is generally known that the first and second kinds of Lucas sequences are associated with the identity

$$V_n^2(P, Q) = DU_n^2(P, Q) + 4Q^n, \tag{3}$$

where the discriminant of the Lucas sequences is $D = P^2 - 4Q$. With respect to these sequences, we have their characteristics the polynomial is as follows:

$$x^2 - Px + Q,$$

such that

$$\alpha = \frac{P + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{D}}{2}$$

are roots of the latter polynomial, where $\alpha.\beta = Q$, $\alpha + \beta = P$, and $\alpha - \beta = \sqrt{D}$. If α/β is not a root of unity, thus, the sequences $\{U_n(P, Q)\}$ and $\{V_n(P, Q)\}$ are called nondegenerate sequences, and degenerate otherwise. Since $gcd(P, Q) = 1$, $P \neq 0$, and $Q \neq 0$, we get that $(P, Q) = \{(-2, 1), (-1, 1), (1, 1), (2, 1)\}$. Consequently, they are sequences that are degenerate if $D = -3$ or $D = 0$. Moreover, these sequences may be expressed in formulas referred to as Binet's formulas, which are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for} \quad n \geq 0, \tag{4}$$

and

$$V_n = \alpha^n + \beta^n \quad \text{for} \quad n \geq 0. \tag{5}$$

For additional information on the Lucas sequences $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$, and their characteristics, we refer to (Chapter 1 in [13]) and [12].

In fact, the study of the solutions of the certain diophantine equations in the Lucas sequences or certain linear recurrences have been in interest to many authors. For example, let's consider the well know Markoff equation

$$x^2 + y^2 + z^2 = 3xyz,$$

that introduced by Markoff in $1879 - 1880$ [10, 11]. The solutions of this equation and some of its generalizations in some liner recurrence sequences were studied by many authors such as Hashim and Tengely [7], and Lucas and Srinivasan [9].

Other problems have been also investigated by several others. For instance, in 1953, Stancliff [14] found the property that $F_{11} = 89$ in the sequence $\{F_n\}$ namely

$$\frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

In 1995, DeWeger [3] determined all the nonnegative integers x and n with $x \geq 2$ for which

$$\frac{1}{F_n} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{x^k}.$$

In 2015, Tengely [15] determined all the nonnegative integers n and x with $x \geq 2$ for which

$$\frac{1}{U_n(P,Q)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k},$$

for certain pairs (P,Q). In 2018, Hashim and Tengely [5] studied the solutions (x,n) with $x \geq 2$ of the equation

$$\frac{1}{U_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k},$$

for certain pairs of (P_1,Q_1) and (P_2,Q_2) with $(P_1,Q_1) \neq (P_2,Q_2)$. In 2021, Hashim [6] studied the solution (x,n) with $x \geq 2$ of the equation

$$\frac{1}{V_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{V_{k-1}(P_1,Q_1)}{x^k},$$

for certain parameters of P_1, Q_1, P_2 and Q_2 with $(P_1,Q_1) = (P_2,Q_2)$ and $(P_1,Q_1) \neq (P_2,Q_2)$.

In this paper, we extend the pervious results by studying the integer solutions (x,n) of the equation

$$\pm\frac{1}{V_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k}, \tag{6}$$

where $-3 \leq P_1, P_2 \leq 3, -2 \leq Q_1 \leq 2$, and $-1 \leq Q_2 \leq 1$. Furthermore, in this study, we investigate the solutions of the latter equation in both cases where the Lucas sequences are nondegenerate and degenerate, with either $(P_1,Q_1) = (P_2,Q_2)$ and $(P_1,Q_1) \neq (P_2,Q_2)$.

We remark that in the proof of our main result, we mainly depends on the following lemmas. Lemma 1.1 is due the result of Köhler [8], and Lemma 1.2 is due to Hashim [6].

Lemma 1.1. *Let a, b, r_0 and r_1 be any complex number. Suppose that the binary sequence $\{r_n\}$ is defined by the recurrence relation $r_{n+1} = ar_n + br_{n-1}$, then*

$$\sum_{k=1}^{\infty} \frac{r_{k-1}}{x^k} = \frac{r_0 x - ar_0 + r_1}{x^2 - ax - b}, \tag{7}$$

where all values of $|x| > |c|$, where c represents a root of $x^2 - ax - b$.

Lemma 1.2. *Suppose that n is a non-negative integer. If $(P, Q) \in \{(2, 1), (-1, 1), (1, 1), (-2, 1)\}$, the degenerate Lucas sequences of the second kind can be represented as the following:*

$$V_n(2, 1) = 2, \tag{8}$$

$$V_n(-1, 1) = \begin{cases} 2 & \text{if} \quad 3|n, \\ -1 & \text{if} \quad 3 \nmid n, \end{cases} \tag{9}$$

$$V_n(1, 1) = \begin{cases} -1 & \text{if} \quad n \equiv \{2, 4\} \pmod 6, \\ 1 & \text{if} \quad n \equiv \{1, 5\} \pmod 6, \\ -2 & \text{if} \quad n = 6t + 3, \quad t \geq 0, \\ 2 & \text{if} \quad n = 6t, \quad t \geq 0, \end{cases} \tag{10}$$

$$V_n(-2, 1) = \begin{cases} 2 & \text{if} \quad n = 2r, \quad r \geq 0\,, \\ -2 & \text{if} \quad n = 2r + 1, \quad r \geq 0. \end{cases} \tag{11}$$

2 Main Approaches

Here, we present the ideas of the main approaches used to prove our main results. In fact, we mainly used two ways to prove the results regarding the chosen values of (P_2, Q_2) for which they are nondegenerate or degenerate. The summary of these approaches is as follows:

- `Elliptic Curves Approach:` That would be done by firstly considering Equation (6) and simplifying it using Equation (7). In other words, we obtain that

$$\pm \frac{1}{V_n(P_2, Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k} = \frac{1}{x^2 - P_1 x + Q_1}. \tag{12}$$

  Hence, we get that

$$V_n(P_2, Q_2) = \pm(x^2 - P_1 x + Q_1). \tag{13}$$

  Now, by multiplying both sides of identity (3) by D, we get that

$$Y^2 = DV_n^2(P_2, Q_2) - 4DQ_2^n, \tag{14}$$

  where $D = P_2^2 - 4Q_2$ and $Y = DU_n(P_2, Q_2)$. Next, by plugging (13) in (14), then we obtain the curve

$$Y^2 = (x^2 - P_1 x + Q_1)^2 D - 4DQ_2^n. \tag{15}$$

  Then, by using the Magma software [1] with the algorithm `IntegralQuarticPoints`, we can solve the elliptic curves of the form (15) for all $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$, and $-1 \leq Q_2 \leq 1$.

- `Quadratic Equations Approach:` Here, we solve Equation (6) after transfering it to equation of the form (13), which represents a simple quadratic equation in the case of (P_2, Q_2) presenting a degenerate Lucas sequence of the second kind using the results of Lemma 1.2.

3 Main Results

Theorem 3.1. *Let $\{U_n(P_1, Q_1)\}$ and $\{V_n(P_2, Q_2)\}$ be the nondegenerate Lucas sequences of the first and second kinds, respectively. If $n \geq 0$, $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$, and $-1 \leq Q_2 \leq 1$, then the integer solutions (x, n) of Equation (6) that satisfy the condition of Lemma 1.1 are given in Table 1:*

Table 1: Solutions of equation (6) with corresponding values of P_1, Q_1, P_2 and Q_2.

(P_1, Q_1)	(P_2, Q_2)	(x, n)
$(-3, -2)$	$(\pm 3, -1)$	$(-4, 0)$
$(-3, -2)$	$(\pm 3, 1)$	$(-4, 0)$
$(-3, -2)$	$(-2, -1)$	$(-4, 0), (-4, 1)$
$(-3, -2)$	$(2, -1)$	$(-4, 0)$
$(-3, -2)$	$(\pm 1, -1)$	$(-4, 0)$
$(-3, -1)$	$(\pm 3, -1)$	$(-67, 7), (-4, 1), (64, 7)$
$(-3, -1)$	$(\pm 3, 1)$	$(-4, 1)$
$(-3, -1)$	$(\pm 1, -1)$	$(-4, 2)$
$(-3, 1)$	$(\pm 3, -1)$	$(-5, 2)$
$(-3, 1)$	$(\pm 1, -1)$	$(-7, 7), (-5, 5), (-3, 1), (4, 7)$
$(-3, 2)$	$(\pm 3, -1)$	$(-3, 0)$
$(-3, 2)$	$(\pm 3, 1)$	$(-3, 0)$
$(-3, 2)$	$(\pm 2, -1)$	$(-4, 2), (-3, 0)$
$(-3, 2)$	$(\pm 1, -1)$	$(-3, 0)$
$(-2, -1)$	$(\pm 3, -1)$	$(-1299, 12), (-120, 8), (-12, 4), (-3, 0), (10, 4), (118, 8), (1297, 12)$
$(-2, -1)$	$(\pm 3, 1)$	$(-5779, 18), (-2208, 16), (-844, 14), (-323, 12), (-124, 10), (-48, 8), (-19, 6), (-8, 4), (-4, 2), (-3, 0), (6, 4), (17, 6), (46, 8), (122, 10), (321, 12), (824, 14), (2206, 16), (5777, 18)$
$(-2, -1)$	$(\pm 2, -1)$	$(-1155, 16), (-199, 12), (-35, 8), (-7, 4), (-5, 3), (-3, 0), (-3, 1), (3, 3), (5, 4), (33, 8), (197, 12), (1153, 16)$
$(-2, -1)$	$(\pm 1, -1)$	$(-5779, 36), (-2208, 32), (-844, 28), (-323, 24), (-124, 20), (-48, 16), (-19, 12), (-8, 8), (-4, 4), (-3, 0), (6, 8), (17, 12), (46, 16), (122, 20), (321, 24), (824, 28), (2206, 32), (5777, 36)$
$(-1, -2)$	$(\pm 3, 1)$	$(-5, 3), (4, 3)$
$(-1, -2)$	$(\pm 1, -1)$	$(-5, 6), (-3, 3), (4, 6)$
$(-1, -1)$	$(\pm 3, -1)$	$(-4, 2), (3, 2)$
$(-1, -1)$	$(\pm 1, -1)$	$(-6, 7), (-4, 5), (-2, 1), (3, 5), (5, 7)$
$(-1, 2)$	$(\pm 2, -1)$	$(-4, 3), (3, 3)$
$(-1, 2)$	$(\pm 1, -1)$	$(-2, 3)$
$(1, -2)$	$(\pm 3, 1)$	$(-4, 3), (5, 3)$
$(1, -2)$	$(\pm 1, -1)$	$(-4, 6), (3, 3), (5, 6)$
$(1, -1)$	$(\pm 3, -1)$	$(-3, 2), (4, 2)$
$(1, -1)$	$(\pm 1, -1)$	$(-5, 7), (-3, 5), (2, 1), (4, 5), (6, 7)$
$(1, 2)$	$(\pm 2, -1)$	$(-3, 3), (4, 3)$
$(1, 2)$	$(\pm 1, -1)$	$(2, 3)$
$(2, -1)$	$(\pm 3, -1)$	$(-1297, 12), (-118, 8), (-10, 4), (3, 0), (12, 4), (120, 8), (1299, 12)$
$(2, -1)$	$(\pm 3, 1)$	$(-5777, 18), (-2206, 16), (-842, 14), (-321, 12), (-122, 10), (-46, 8), (-17, 6), (-6, 4), (3, 0), (4, 2), (8, 4), (19, 6), (48, 8), (124, 10), (323, 12), (844, 14), (2208, 16), (5779, 18)$

Table 1: Solutions of equation (6) with corresponding values of P_1, Q_1, P_2 and Q_2 (continued).

$(2,-1)$	$(\pm 2,-1)$	$(-1153,16),(-197,12),(-33,8),(-5,4),(-3,3),(3,0),$ $(3,1),(5,3),(7,4),(35,8),(199,12),(1155,16)$
$(2,-1)$	$(\pm 1,-1)$	$(-5777,36),(-2206,32),(-842,28),(-321,24),(-122,20),$ $(-46,16),(-17,12),(-6,8),(3,0),(4,4),(8,8),(19,12),$ $(48,16),(124,20),(323,24),(844,28),(2208,32),(5779,36)$
$(3,-2)$	$(\pm 3,-1)$	$(4,0)$
$(3,-2)$	$(\pm 3,1)$	$(4,0)$
$(3,-2)$	$(\pm 2,-1)$	$(4,0)$
$(3,-2)$	$(\pm 1,-1)$	$(4,0)$
$(3,-1)$	$(\pm 3,-1)$	$(-64,7),(4,1),(67,7)$
$(3,-1)$	$(\pm 3,1)$	$(4,1)$
$(3,-1)$	$(\pm 1,-1)$	$(4,2)$
$(3,1)$	$(\pm 3,-1)$	$(5,2)$
$(3,1)$	$(\pm 1,-1)$	$(-4,7),(3,1),(5,5),(7,7)$
$(3,2)$	$(\pm 3,-1)$	$(3,0)$
$(3,2)$	$(\pm 3,1)$	$(3,0)$
$(3,2)$	$(\pm 2,-1)$	$(3,0),(4,2)$
$(3,2)$	$(\pm 1,-1)$	$(3,0)$

Proof. To prove the theorem, here we used the Elliptic curves approach that is described in Section 2. As stated in the theorem, here we deal with nondegenerate Lucas sequences. First, we seek the value of x of the solutions (x,n) of Equation (6), so that, $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$, and $-1 \leq Q_2 \leq 1$, where x satisfies the condition stated in Lemma 1.1. In other words, to obtain the x values, we substitute in Equation (15) the values of (P_1, Q_1) and (P_2, Q_2) with $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$, and $-1 \leq Q_2 \leq 1$ to obtain the corresponding elliptic curves that can be solved using the Magma software with the algorithm `IntegralQuarticPoints`. Indeed, we compute the solutions of Equation (6) in details for certain values of (P_1, Q_1) and (P_2, Q_2), and the others will be achieved similarly. For example, if $(P_1, Q_1) = (-3,-2)$ and $(P_2, Q_2) = (\pm 3,-1)$ we substitute these values into the Equation (15) to get the elliptic curves $Y^2 = 13x^4 + 78x^3 + 65x^2 - 156x + 104$ and $Y^2 = 13x^4 + 78x^3 + 65x^2 - 156x$. Then, by using the Magma software function [`IntegralQuarticPoints([13,78,65,-156,0]);`] for the latter equation. Then, we just get $x \in \{-4,-3,0,1\}$. We look for x values for which $|x| > |-3.561|$, where $|-3.561|$ represents the approximated maximum absolute value for the solutions of x^2+3x-2 as required by Lemma 1.1. Then, we only consider $x \in \{-4\}$. Similarly, if $(P_1, Q_1) = (1,2)$ and $(P_2, Q_2) = (-2,-1)$, then we only get $x \in \{-3,4\}$. Likewise, if $(P_1, Q_1) = (P_2, Q_2) = (3,-1)$, then we get $x \in \{-64,4,67\}$. Similarly, in the case of $(P_1, Q_1) = (-1,-2)$ and $(P_2, Q_2) = (1,-1)$, when we substitute the relevant given values into Equation (15), we obtain the values of $x \in \{-5,-3,-1,0,2,4\}$. The values of x must satisfy the condition of Lemma 1.1 so that, $|x| > |-2|$, where $|-2|$ represents the absolute value of the maximum root of the quadratic equation $x^2 + x - 2$. Therefore, the only values that satisfy the condition of Lemma 1.1 are $x \in \{-5,-3,4\}$.

Furthermore, we list in Table 2 all the values of the x-coordinates of the solutions (x,Y) of the elliptic curve Equations (15) corresponding to the remaining values of P_1, Q_1, P_2 and Q_2 with $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$ and $-1 \leq Q_2 \leq 1$. Note that we here list all the x values with $|x| > |r|$ with r represents a root of $x^2 - P_1x + Q_1$.

Table 2: Values of x satisfying the elliptic curves $Y^2 = (x^2 - P_1x + Q_1)^2D - 4DQ_2^n$.

(P_1, Q_1)	(P_2, Q_2)	x	(P_1, Q_1)	(P_2, Q_2)	x
$(-3,-2)$	$(2,-1)$	-4	$(-3,-2)$	$(-2,-1)$	-4
$(-3,-2)$	$(\pm3,1)$	-4	$(-3,-2)$	$(\pm1,-1)$	-4
$(-3,-1)$	$(\pm3,-1)$	$-67,-4,64$	$(-3,-1)$	$(\pm2,-1)$	$-$
$(-3,-1)$	$(\pm3,1)$	-4	$(-3,-1)$	$(\pm1,-1)$	-4
$(-3,1)$	$(\pm3,-1)$	-5	$(-3,1)$	$(\pm3,1)$	$-$
$(-3,1)$	$(\pm1,-1)$	$-7,-5,-3,4$	$(-3,1)$	$(\pm2,-1)$	$-$
$(-3,2)$	$(\pm3,-1)$	-3	$(-3,2)$	$(\pm2,-1)$	$-4,-3$
$(-3,2)$	$(\pm3,1)$	-3	$(-3,2)$	$(\pm1,-1)$	-3
$(-2,-1)$	$(\pm3,-1)$	$-1299,-120,-12,-3,10,118,1297$	$(-2,-1)$	$(\pm3,1)$	$-5779,-2208,-844,-323,-124,-48,-19,-8,-4,-3,6,17,46,122,321,842,2206,5777$
$(-2,-1)$	$(\pm2,-1)$	$-1155,-199,-35,-7,-5,-3,3,5,33,197,1153$	$(-2,-1)$	$(\pm1,-1)$	$-5779,-2208,-844,-323,-124,-48,-19,-8,-4,-3,6,17,46,122,321,842,2206,5777$
$(-1,-2)$	$(\pm3,-1)$	$-$	$(-1,-2)$	$(\pm2,-1)$	$-$
$(-1,-2)$	$(\pm3,1)$	$-5,4$	$(-1,-2)$	$(-1,-1)$	$-5,-3,4$
$(-1,-1)$	$(\pm3,-1)$	$-4,3$	$(-1,-1)$	$(\pm2,-1)$	$-$
$(-1,-1)$	$(\pm3,1)$	$-$	$(-1,-1)$	$(\pm1,-1)$	$-6,-4,-2,3,5$
$(-1,2)$	$(\pm3,-1)$	$-$	$(-1,2)$	$(\pm2,-1)$	$-4,3$
$(-1,2)$	$(\pm3,1)$	$-$	$(-1,2)$	$(\pm1,-1)$	-2
$(1,-2)$	$(\pm3,-1)$	$-$	$(1,-2)$	$(\pm2,-1)$	$-$
$(1,-2)$	$(\pm3,1)$	$-4,5$	$(1,-2)$	$(\pm1,-1)$	$-4,3,5$
$(1,-1)$	$(\pm3,-1)$	$-3,4$	$(1,-1)$	$(\pm2,-1)$	$-$
$(1,-1)$	$(\pm3,1)$	$-$	$(1,-1)$	$(\pm1,-1)$	$-5,-3,2,4,6$
$(1,2)$	$(\pm3,-1)$	$-$	$(1,2)$	$(2,-1)$	$-3,4$
$(1,2)$	$(\pm3,1)$	$-$	$(1,2)$	$(\pm1,-1)$	2
$(2,-1)$	$(\pm3,-1)$	$-1297,-118,-10,3,12,120,1299$	$(2,-1)$	$(\pm3,1)$	$-5777,-2206,-842,-321,-122,-46,-17,-6,3,4,8,19,48,124,323,844,2208,5779$
$(2,-1)$	$(\pm2,-1)$	$-1153,-197,-33,-5,-3,3,5,7,35,199,1155$	$(2,-1)$	$(\pm1,-1)$	$-5777,-2206,-842,-321,-122,-46,-17,-6,3,4,8,19,48,124,323,844,2208,5779$
$(3,-2)$	$(\pm3,-1)$	4	$(3,-2)$	$(\pm2,-1)$	4
$(3,-2)$	$(\pm3,1)$	4	$(3,-2)$	$(\pm1,-1)$	4
$(3,-1)$	$(-3,-1)$	$-64,4,67$	$(3,-1)$	$(\pm2,-1)$	$-$
$(3,-1)$	$(\pm3,1)$	4	$(3,-1)$	$(\pm1,-1)$	4
$(3,1)$	$(\pm3,-1)$	5	$(3,1)$	$(\pm2,-1)$	$-$
$(3,1)$	$(\pm3,1)$	$-$	$(3,1)$	$(\pm1,-1)$	$-4,3,5,7$
$(3,2)$	$(\pm3,-1)$	3	$(3,2)$	$(\pm2,-1)$	$3,4$
$(3,2)$	$(\pm3,1)$	3	$(3,2)$	$(\pm1,-1)$	3

Finally, we search for respective n values from Equation (13) after substituting the obtained values of x and the corresponding pairs (P_1, Q_1) and (P_2, Q_2). Indeed, we calculate the values of n for some cases, and the others are approached in the same way. For example, in the case of $(P_1, Q_1) = (-3, -2)$ and $(P_2, Q_2) = (\pm3, -1)$, we have that $x \in \{-4\}$. If $x = -4$, then Equation (13) gives that

$$V_n(\pm3, -1) = \pm((-4)^2 + (3)(-4) - 2) = \pm2.$$

If $V_n(\pm 3, -1) = -2$, then there is no such value for n. On the other hand, if $V_n(\pm 3, -1) = 2$, then we have $n = 0$. Hence, we obtain the desired solution $(x, n) = (-4, 0)$. Next, the other case is $(P_1, Q_1) = (1, 2)$ and $(P_2, Q_2) = (-2, -1)$, and here we have $x \in \{-3, 4\}$. If $x = -3, 4$ respectively, Equation (13) implies that

$$V_n(-2, -1) = \pm((-3)^2 + (3) + 2) = \pm 14$$

and

$$V_n(-2, -1) = \pm(4^2 - 4 + 2) = \pm 14.$$

Hence, $V_n(-2, -1) = -14$ gives $n = 3$. Thus, the intended solution $(x, n) = \{(-3, 3), (4, 3)\}$ is obtained. However, if $V_n(-2, -1) = 14$, then no such value for n. Next, we consider $(P_1, Q_1) = (P_2, Q_2) = (3, -1)$, and here we have that $x \in \{-64, 4, 67\}$. We see that for $x = -64$, Equation (13) leads to

$$V_n(3, -1) = \pm((-64)^2 + 3(64) - 1) = \pm 4287.$$

If $V_n(3, -1) = 4287$, then $n = 7$. Hence, the desired solution is obtained. But for $V_n(3, -1) = -4287$ there is not such value for n. Similarly, in the case of $x = 4$ we obtain that

$$V_n(3, -1) = \pm(4^2 - 3(4) - 1) = \pm 3.$$

In fact, we have a value for n only in the case of $V_n(3, -1) = 3$, which gives $n = 1$. So, we have $(x, n) = (4, 1)$. In the case of $x = 67$, we obtain that

$$V_n(3, -1) = \pm(67^2 - 3(67) - 1) = \pm 4287.$$

We get a value for n only when $V_n(3, -1) = 4287$, which is $n = 7$. Consequently, we obtain the solution $(x, n) = (67, 7)$. Next, we consider $(P_1, Q_1) = (-1, -2)$ and $(P_2, Q_2) = (1, -1)$ we have that $x \in \{-5, -3, 4\}$. If $x = -5$ and $x = 4$ respectively, we substitute these values in Equation (13) to get

$$V_n(1, -1) = \pm((-5)^2 - 5 - 2) = \pm 18$$

and

$$V_n(1, -1) = \pm(4^2 + 4 - 2) = \pm 18.$$

If $V_n(1, -1) = 18$, then we have $n = 6$. Hence, the required solution is $(x, n) \in \{(-5, 6), (4, 6)\}$. But $V_n(1, -1) = -18$ is not Lucas number of the secund kind at $(P_2, Q_2) = (1, -1)$. Hence, there is not such values for n. Similarly, if $x = -3$ we obtain that

$$V_n(1, -1) = \pm((-3)^2 - 3 - 2) = \pm 4.$$

We get $n = 3$ only when $V_n(1, -1) = 4$. Therefore, the required solution is $(x, n) = (-3, 3)$. The remaining cases can be approached in a similar manner as previous cases, therefore we omit the details of computations. □

Theorem 3.2. *Suppose that $\{U_n(P_1, Q_1)\}$ and $\{V_n(P_2, Q_2)\}$ are respectively the Lucas sequences of the first and second kind with $n \geq 0$, $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$ and $-1 \leq Q_2 \leq 1$. If at least one of these sequences is degenerate, then the integer solutions (x, n) of Equation (6) that satisfy the condition of Lemma 1.1 are given in Table 3:*

Table 3: Solutions of equation (6) with degenerate Lucas sequences.

(P_1, Q_1)	(P_2, Q_2)	x	$\{n\}$
$(-3,-2)$	$(2,1)$	-4	$\{n : n \geq 0\}$
$(-3,2)$	$(2,1)$	-3	$\{n : n \geq 0\}$
$(-2,-1)$	$(2,1)$	-3	$\{n : n \geq 0\}$
$(2,-1)$	$(2,1)$	3	$\{n : n \geq 0\}$
$(3,-2)$	$(2,1)$	4	$\{n : n \geq 0\}$
$(3,2)$	$(2,1)$	3	$\{n : n \geq 0\}$
$(-3,-2)$	$(-1,1)$	-4	$\{n : 3 \mid n\}$
$(-3,1)$	$(-1,1)$	-3	$\{n : 3 \nmid n\}$
$(-3,2)$	$(-1,1)$	-3	$\{n : 3 \mid n\}$
$(-2,-1)$	$(-1,1)$	-3	$\{n : 3 \mid n\}$
$(-2,1)$	$(-1,1)$	-2	$\{n : 3 \nmid n\}$
$(-1,-1)$	$(-1,1)$	-2	$\{n : 3 \nmid n\}$
$(1,-1)$	$(-1,1)$	2	$\{n : 3 \nmid n\}$
$(2,-1)$	$(-1,1)$	3	$\{n : 3 \mid n\}$
$((2,1)$	$(-1,1)$	2	$\{n : 3 \nmid n\}$
$(3,-2)$	$(-1,1)$	4	$\{n : 3 \mid n\}$
$(3,1)$	$(-1,1)$	3	$\{n : 3 \nmid n\}$
$(3,2)$	$(-1,1)$	3	$\{n : 3 \mid n\}$
$(-3,-2)$	$(1,1)$	-4	$\{n : n = 6t \text{ and } n = 6t+3,\ t \geq 0\}$
$(-3,1)$	$(1,1)$	-3	$\{n : n \equiv \{1,5\} \pmod 6 \text{ and } n \equiv \{2,4\} \pmod 6\}$
$(-3,2)$	$(1,1)$	-3	$\{n : n = 6t \text{ and } n = 6t+3,\ t \geq 0\}$
$(-2,-1)$	$(1,1)$	-3	$\{n : n = 6t \text{ and } n = 6t+3,\ t \geq 0\}$
$(-2,1)$	$(1,1)$	-2	$\{n : n \equiv \{1,5\} \pmod 6 \text{ and } n \equiv \{2,4\} \pmod 6\}$
$(-1,-1)$	$(1,1)$	-2	$\{n : n \equiv \{1,5\} \pmod 6 \text{ and } n \equiv \{2,4\} \pmod 6\}$
$(1,-1)$	$(1,1)$	2	$\{n : n \equiv \{1,5\} \pmod 6 \text{ and } n \equiv \{2,4\} \pmod 6\}$
$(2,-1)$	$(1,1)$	3	$\{n : n = 6t \text{ and } n = 6t+3,\ t \geq 0\}$
$(2,1)$	$(1,1)$	2	$\{n : n \equiv \{1,5\} \pmod 6 \text{ and } n \equiv \{2,4\} \pmod 6\}$
$(3,-2)$	$(1,1)$	4	$\{n : n = 6t \text{ and } n = 6t+3,\ t \geq 0\}$
$(3,1)$	$(1,1)$	3	$\{n : n \equiv \{1,5\} \pmod 6 \text{ and } n \equiv \{2,4\} \pmod 6\}$
$(3,2)$	$(1,1)$	3	$\{n : n = 6t \text{ and } n = 6t+3,\ t \geq 0\}$
$(-3,-2)$	$(-2,1)$	-4	$\{n : n = 2r \text{ and } n = 2r+1,\ r \geq 0\}$
$(-3,2)$	$(-2,1)$	-3	$\{n : n = 2r \text{ and } n = 2r+1,\ r \geq 0\}$
$(-2,-1)$	$(-2,1)$	-3	$\{n : n = 2r \text{ and } n = 2r+1,\ r \geq 0\}$
$(2,-1)$	$(-2,1)$	3	$\{n : n = 2r \text{ and } n = 2r+1,\ r \geq 0\}$
$(3,-2)$	$(-2,1)$	4	$\{n : n = 2r \text{ and } n = 2r+1,\ r \geq 0\}$
$(3,2)$	$(-2,1)$	3	$\{n : n = 2r \text{ and } n = 2r+1,\ r \geq 0\}$
$(2,1)$	$(\pm 3,-1)$	$-5,7$	$\{3\}$
$(2,1)$	$(\pm 1,-1)$	2 3	$\{1\}$ $\{3\}$
$(-1,1)$	$(\pm 3,-1)$	-2	$\{1\}$
$(-1,1)$	$(\pm 3,1)$	$-3,2$ -2	$\{2\}$ $\{1\}$

Table 3: Solutions of equation (6) with degenerate Lucas sequences (continued).

$(-1,1)$	$(\pm1,-1)$	$-3,2$ -2	$\{4\}$ $\{2\}$
$(1,1)$	$(\pm3,-1)$	2	$\{1\}$
$(1,1)$	$(\pm3,1)$	$-2,3$ 2	$\{2\}$ $\{1\}$
$(1,1)$	$(\pm1,-1)$	$-2,3$ 2	$\{4\}$ $\{2\}$
$(-2,1)$	$(\pm3,-1)$	$-7,5$	$\{3\}$
$(-2,1)$	$(\pm1,-1)$	-3 -2	$\{3\}$ $\{1\}$

Proof. We divide the proof of the theorem into two cases regarding the values of (P_2, Q_2) in which they are either degenerate or nondegenerate.

- `Degenerate Sequences Case.` Here, we use the Quadratic Equations Approach, that is described in Section 2. In this case, we respectively deal with the degenerate and nondegenerate Lucas sequences concerning the values of (P_1, Q_1) with either $(P_1, Q_1) = (P_2, Q_2)$ and $(P_1, Q_1) \neq (P_2, Q_2)$. We solve the equations in the case of $(P_2, Q_2) \in \{(2, 1), (-1, 1), (1, 1), (-2, 1)\}$ with all (P_1, Q_1) where $-3 \leq P_1 \leq 3$ and $-2 \leq Q_1 \leq 2$. Firstly, we deal with the cases of $(P_1, Q_1) = (P_2, Q_2)$. Let's consider $(P_1, Q_1) = (P_2, Q_2) = (-1, 1)$, then Equation (13) gives that
$$V_n(-1, 1) = \pm(x^2 + x + 1). \tag{16}$$
From Lemma 1.1, we note that the values of x for the above equation in which $|x| > |r|$ with r represents a maximum root of $x^2 - P_1x + Q_1 = x^2 + x + 1$, that is $|x| > 1$.

  From Equation (9), we have that $V_n(-1, 1) = -1, 2$. If $V_n(-1, 1) = -1$, then Equation (16) implies that
$$x^2 + x + 1 \pm 1 = 0.$$
Thus, we get $x = -1, 0$. Since have the absolute values of the integer solutions of the quadratic equation are not satisfying the condition of Lemma 1.1, so we ignore these solutions. Similarly, if we consider $V_n(-1, 1) = 2$, then there are no integer values for x. The remaining pairs of the above set are treated similarly.

  Secondly, we deal with the cases of $(P_1, Q_1) \neq (P_2, Q_2)$, where (P_2, Q_2) are degenerate. Suppose that $(P_2, Q_2) = (1, 1)$, then Equation (13) leads to
$$V_n(1, 1) = \pm(x^2 - P_1x + Q_1). \tag{17}$$
From Lemma 1.2, Equation (10) gives that $V_n(1, 1) = (-1)^n, 2(-1)^n$ for certain values of n with $n \geq 0$. If $V_n(1, 1) = \pm1$, then Equation (17) gives that
$$x^2 - P_1x + (Q_1 \pm 1) = 0. \tag{18}$$
This equation can be solved easily for all $-3 \leq P_1 \leq 3$ and $-2 \leq Q_1 \leq 2$, and the corresponding values of n can be followed from substituting the obtained values of x, that satisfy the condition of Lemma 1.1, in Equation (17). If we assume that $(P_1, Q_1) = (-1, 2)$ when substituting into Equation (18) we get the equations $x^2 + x + 3 = 0$ and $x^2 + x + 1 = 0$. Therefore, those equations have no integer solution. For certain values of P_1 and Q_1, in Table 4 we indicate the integer solutions of Equation (18) and their corresponding values of n.

Table 4: Solutions of equation (18) for certain values of P_1 and Q_1.

(P_1, Q_1)	x	$\{n\}$
$(-2, 1)$	-2	$\{n : n \equiv \{1, 5\} \pmod 6$ and $n \equiv \{2, 4\} \pmod 6\}$
$(2, 1)$	2	$\{n : n \equiv \{1, 5\} \pmod 6$ and $n \equiv \{2, 4\} \pmod 6\}$
$(-3, 1)$	-3	$\{n : n \equiv \{1, 5\} \pmod 6$ and $n \equiv \{2, 4\} \pmod 6\}$
$(-3, -1)$	$-$	$-$

But for

$$\pm 2 = V_n(1, 1) = \pm(x^2 - P_1 x + Q_1), \tag{19}$$

with $(P_1, Q_1) \in \{(-2, 1), (2, 1), (-3, 1), (-3, -1)\}$, the equation has no integer solution. However, for $(P_1, Q_1) = (-1, 2)$, then Equation (19) has the integer solutions $x = -1, 0$, which doesn't satisfy the condition of Lemma 1.1 with which the approximated maximum absolute value of the roots of $x^2 - P_1 x + Q_1 = x^2 + x + 2$ is 1.414. Thus, we have no solution in this case. In fact, the other cases are treated similarly.

- `Nondegenerate sequences Case.` In this case, we used the Elliptic curve approach (presented in Section 2) with which the Lucas sequences are the degenerate and nonodegenerate with respect to (P_1, Q_1) and (P_2, Q_2), respectively. The initial action is to locate the value of x of solutions (x, n) of Equation (6) such that $(P_1, Q_1) \in \{(2, 1), (-1, 1), (1, 1), (-2, 1)\}$, $-3 \le P_2 \le 3$, and $-1 \le Q_2 \le 1$ so that $(P_2, Q_2) \notin \{(2, 1), (-1, 1), (1, 1), (-2, 1)\}$ where x satisfies the condition stated in Lemma 1.1. In fact, to obtain the values of x we substitute in Equation (15) the values of (P_1, Q_1) and (P_2, Q_2) to get the corresponding elliptic curves that can be solved using the Magma software with the algorithm `IntegralQuarticPoints`. In fact, we calculate the solutions of Equation (6) in details for certain values of P_1, Q_1, P_2 and Q_2, The others will be achieved in the same way. For instance, if $(P_1, Q_1) = (2, 1)$ and $(P_2, Q_2) = (3, -1)$ we then substitute these values into the Equation (15) we only get $x \in \{-5, 7\}$, which satisfies the condition of Lemma 1.1 as it's greater that the absolute value of the roots of $x^2 - P_1 x + Q_1 = x^2 - 2x + 1$. Similarly, if $(P_1, Q_1) = (-2, 1)$ and $(P_2, Q_2) = (-1, -1)$ we obtain that $x \in \{-3, -2\}$.

  Moreover, Table 5 we list all the values of the x-coordinates of the solutions (x, Y) of the elliptic curve Equations (15) concerning to the remaining values of P_1, Q_1, P_2 and Q_2 with $(P_1, Q_1) \in \{(2, 1), (-1, 1), (1, 1), (-2, 1)\}$, $-3 \le P_2 \le 3$ and $-1 \le Q_2 \le 1$ so that $(P_2, Q_2) \notin \{(2, 1), (-1, 1), (1, 1), (-2, 1)\}$. Consequently, we list all the values of x with $|x| > |r|$, where r is the root of $x^2 - P_1 x + Q_1$.

Table 5: Values of x satisfying the elliptic curves $Y^2 = (x^2 - P_1 x + Q_1)^2 D - 4DQ_2^n$.

(P_1, Q_1)	(P_2, Q_2)	x	(P_1, Q_1)	(P_2, Q_2)	x
$(2, 1)$	$(-3, -1)$	$-5, 7$	$(2, 1)$	$(\pm 3, 1)$	$-$
$(2, 1)$	$(\pm 2, -1)$	$-$	$(2, 1)$	$(\pm 1, -1)$	$2, 3$
$(-1, 1)$	$(\pm 3, -1)$	-2	$(-1, 1)$	$(\pm 3, 1)$	$-3, -2, 2$
$(-1, 1)$	$(\pm 2, -1)$	$-$	$(-1, 1)$	$(\pm 1, -1)$	$-3, -2, 2$
$(1, 1)$	$(\pm 3, -1)$	2	$(1, 1)$	$(\pm 3, 1)$	$-2, 2, 3$
$(1, 1)$	$(\pm 2, -1)$	$-$	$(1, 1)$	$(\pm 1, -1)$	$-2, 2, 3$
$(-2, 1)$	$(\pm 3, -1)$	$-7, 5$	$(-2, 1)$	$(\pm 3, 1)$	$-$
$(-2, 1)$	$(\pm 2, -1)$	$-$	$(-2, 1)$	$(1, -1)$	$-3, -2$

Finally, we determine the respective n values from Equation (13) where we substitute the obtained x values and the related parameters P_1, Q_1, P_2 and Q_2. In fact, we compute the values of n in certain instances, and the others are treated in a similar way. For example, if $(P_1, Q_1) = (2, 1)$ and $(P_2, Q_2) = (3, -1)$ we have that $x \in \{-5, 7\}$. If $x = -5$, then Equation (13) implies that

$$V_n(3, -1) = \pm((-5)^2 - 2(-5) + 1) = \pm 36.$$

Also, in the case $x = 7$ we have that

$$V_n(3, -1) = \pm(7^2 - 2(7) + 1) = \pm 36.$$

If $V_n(3, -1) = 36$, then we have $n = 3$. From the other side, if $V_n(3, -1) = -36$, then there is no such value for n. Here, we obtain the required solutions such that $\{(x, n)\} \in \{(-5, 3), (7, 3)\}$.

Next, we consider $(P_1, Q_1) = (-2, 1)$ and $(P_2, Q_2) = (-1, -1)$, and here we have $x \in \{-3, -2\}$. If $x = -3$, then Equation (13) gives that

$$V_n(-1, -1) = \pm((-3)^2 + 2(-3) + 1) = \pm 4.$$

Indeed, we obtain a value for n just when $V_n(-1, -1) = -4$, that is $n = 3$. Hence, we obtain the desired solution $(x, n) = (-3, 3)$. If $x = -2$, then from Equation (13) we obtain that

$$V_n(\pm 1, -1) = \pm((-2)^2 + 2(-2) + 1) = \pm 1.$$

We get a value for n only when $V_n(-1, -1) = -1$ that is given by $n = 1$. Here, we again obtain the required solution $(x, n) = (-2, 1)$. The remaining cases can be approached similarly to the remaining cases, therefore we omit the details of computations.

□

4 Conclusion

This paper mainly focuses on providing a technique for studying the integral solutions (x, n) of the diophantine equation

$$\pm \frac{1}{V_n(P_2, Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k}$$

in the cases of $(P_1, Q_1) \neq (P_2, Q_2)$ and $(P_1, Q_1) = (P_2, Q_2)$. Our major strategy is to convert this equation into a quadratic equation that can be solved using the quadratic formula given by (13) or an elliptic curve equation given by (15) that can be solved with ease using the Magma software. Furthermore, we apply our technique for solving the equation in cases of $-3 \leq P_1, P_2 \leq 3$, $-2 \leq Q_1 \leq 2$, and $-1 \leq Q_2 \leq 1$. In fact, we obtain collections of solutions given in the section of the main results. These results represent interesting representations of reciprocals of the Lucas sequences of the second kind, that are equal to infinite sums of terms in the Lucas sequences of the first kind in base x.

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References

[1] W. Bosma, J. Cannon & C. Playoust (1997). The magma algebra system I: The user language. *Journal of Symbolic Computation*, *24*(3-4), 235–265. https://doi.org/10.1006/jsco.1996.0125.

[2] S. Boudaoud, D. Bellaouar & A. Boudaoud (2020). Nonclassical study on certain diophantine inequalities involving multiplicative arithmetic functions. *Malaysian Journal of Mathematical Sciences*, *14*(1), 17–39.

[3] B. De Weger (1995). A curious property of the eleventh Fibonacci number. *The Rocky Mountain Journal of Mathematics*, *25*(3), 977–994. http://dx.doi.org/10.1216/rmjm/1181072199.

[4] M. J. Deng, J. Guo & A. J. Xu (2018). A note on the diophantine equation $x^2 + (2c-1)^m = c^n$. *Bulletin of the Australian Mathematical Society*, *98*(2), 188–195. https://doi.org/10.1017/S0004972718000369.

[5] H. R. Hashim & S. Tengely (2018). Representations of reciprocals of Lucas sequences. *Miskolc Mathematical Notes*, *19*(2), 865–872. https://doi.org/10.18514/MMN.2018.2520.

[6] H. R. Hashim (2021). Curious properties of generalized Lucas numbers. *Boletín de la Sociedad Matemática Mexicana*, *27*, 1–10. https://doi.org/10.1007/s40590-021-00391-7.

[7] H. R. Hashim & S. Tengely (2020). Solutions of a generalized Markoff equation in Fibonacci numbers. *Mathematica Slovaca*, *70*(5), 1069–1078. https://doi.org/10.1515/ms-2017-0414.

[8] G. Köhler (1985). Generating functions of Fibonacci-like sequences and decimal expansions of some fractions. *The Fibonacci Quarterly*, *23*(1), 29–35.

[9] F. Luca & A. Srinivasan (2018). Markov equation with Fibonacci components. *Fibonacci Quart*, *56*(2), 126–129.

[10] A. Markoff (1879). Sur les formes quadratiques binaires indéfinies. *Mathematische Annalen*, *15*(3), 381–406.

[11] A. Markoff (1880). Sur les formes quadratiques binaires indéfinies. (Second mémoire). *Mathematische Annalen*, *17*(3), 379–399.

[12] Ö. Özer (2017). Fibonacci sequence and continued fraction expansions in real quadratic number fields. *Malaysian Journal of Mathematical Sciences*, *11*(1), 97–118.

[13] P. Ribenboim (2000). *My numbers, my friends: Popular lectures on number theory*. Springer Science & Business Media, New York.

[14] F. Stancliff (1953). A curious property of a_{ii}. *Scripta Mathematica*, *19*, 126.

[15] S. Tengely (2015). On the Lucas sequence equation $\frac{1}{U_n} = \sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}$. *Periodica Mathematica Hungarica*, *71*, 236–242. https://doi.org/10.1007/s10998-015-0101-4.